JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 19, No. 2, March–April 1996

Formulation of Stability Conditions for Systems Containing Driven Rotors

I. Michael Ross*
U.S. Naval Postgraduate School, Monterey, California 93943-5000

The much maligned energy-sink analysis is revisited to derive stability conditions for a dual-spin spacecraft that is subjected to possible energy additions by the motor torque. Landon's old idea of subtracting the work done by the motor torque from the energy function is applied to the case when both the platform and the rotor are quasirigid. This is made possible by way of a separation axiom that allows one to obtain an expression for the motor torque. It is shown that this process leads naturally to a symmetric stability condition expressible in terms of Hubert's core energy.

Nomenclature

E	= kinetic energy of a dual spinner
$E_{c,P}$	= platform core energy
$E_{c,R}$	= rotor core energy
$\dot{E}_D$	= energy dissipation rate due to all damping mechanisms
$\dot{E}_P$	= energy dissipation rate in the platform
$\dot{E}_R$	= energy dissipation rate in the rotor
H	= magnitude of the central angular momentum of a dual-spin spacecraft
H_s	= spin component of the central angular momentum of a dual-spin spacecraft
I_s	= moment of inertia of an axisymmetric dual-spin spacecraft about the spin axis
$I_{s,P}$	= moment of inertia of the platform about the spin axis
I_t	= moment of inertia of an axisymmetric dual-spin spacecraft about the transverse axis
J	= moment of inertia of the rotor about the spin axis
Q	= parameter as defined by Eq. (27b)
$T_{P/M}$	= torque on the platform due to the motor when either the platform or the rotor is rigid
$T^*_{P/M}$	= torque on the platform due to the motor
T_R	= net axial torque on a rigid rotor
$T_{R/M}$	= torque on the rotor due to the motor when either the platform or the rotor is rigid
$T^*_{R/M}$	= torque on the rotor due to the motor
$\dot{W}$	= rate of work done by the motor torque
η	= nutation angle
λ_P	= platform nutation frequency
λ_R	= rotor nutation frequency
Ω	= spin speed of the rotor relative to the platform
ω_s	= component of the inertial angular velocity of the platform along the spin axis of the dual-spin spacecraft
ω_t	= component of the inertial angular velocity of the platform along the transverse axis of the dual-spin spacecraft

Introduction

DESPITE its success in arriving at useful rules of thumb for designing spacecraft, the energy-sink analysis has been viewed with much skepticism and debated for quite some time.[1–10] As was pointed out in Ref. 10, this was, in part, the result of varying interpretations of this method of analysis. At least three problems have been lumped under the heading of energy-sink methods, viz., problem 1, determining the stability of a given equilibrium[1,2]; problem 2, estimating the rate of convergence or divergence from a known equilibrium state[5,10]; and problem 3, finding equilibrium states.[7]

In what follows, we will refer to the energy-sink theory as one that is used for solving any of these problems for unknown or unmodeled mechanisms of energy sink. If the energy-sink mechanism can be accurately modeled, then there is no need for a theory as the system can (and should) be analyzed from the equations of motion. A theory is still useful for understanding broader and perhaps fundamental issues.

Generally speaking, equilibrium states are those that extremize energy, and those that minimize it are said to be stable. In solving problem 2, Kane and Levinson[5] noted that erroneous results could be obtained if the energy-sink theory was extrapolated to systems containing driven rotors that could potentially add energy to the system. They demonstrated this by applying the theory to a dual-spin spacecraft that contained a discrete damper. Thus, they argued, the Landon–Iorillo stability criterion[10–15]

$$(\dot{E}_P/\lambda_P) + (\dot{E}_R/\lambda_R) \leq 0 \tag{1}$$

(for problem 1) will yield incorrect results since in its derivation it is widely assumed [12–15] that

$$\dot{E} = \dot{E}_P + \dot{E}_R \tag{2}$$

and, consequently,

$$\dot{E} \leq 0 \tag{3}$$

Cochran and Shu[8] showed that if the energy additions by the motor were properly accounted for, then an application of the energy-sink theory for problem 2 yields correct results. Their analysis, however, was mostly restricted to a system containing a discrete damper, and they did not explain the apparent validity of the theory despite energy additions. Ross[10] showed that the theory could solve problem 2 if Hubert's[6,7] core energy was used instead of the total energy. He also showed that when the relative spin speed of the rotor of a prolate dual spinner is held a constant, stability requires that energy be maximal, not minimal.[10] Thus, the question remains: What is the stability criteria for systems containing driven rotors? This paper addresses this question (i.e., problem 1), and we show that Eq. (1) remains valid without using the assumptions of Eq. (2) or (3). This is made possible by clarifying a separation axiom and using Newton's third law of motion. This process also leads to an alternative formulation of the stability criterion given in a symmetric form by the use of the core-energy concept.

Definitions and Assumptions

In the following analysis, we will use the term quasirigid body in a slightly more general form than that used by Hughes.[14]

Definition: A quasirigid body is a rigid body plus an energy sink. An energy sink is a device, discrete or distributed, that when placed

Received Aug. 23, 1993; revision March 20, 1995; accepted for publication Aug. 25, 1995.

*Assistant Professor, Department of Aeronautics and Astronautics, Mail Code AA-Ro. Senior Member AIAA.

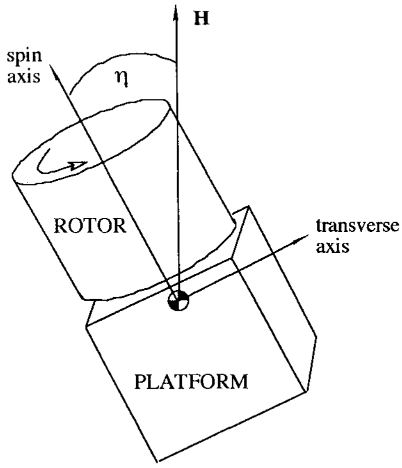

Fig. 1 Dual-spin spacecraft.

on a rigid body has the effect of dissipating kinetic energy during its motion; it does, however, allow equilibrium states (for this quasirigid body) characterized by constant energy. In addition, this device does not make a significant contribution to the rigid body's kinetic energy or angular momentum.

This definition of quasirigidity permits us to use certain rigid-body equations (kinetic energy and angular momentum) as zeroth-order approximations[8] for quite a few systems, such as slightly flexible bodies, rigid bodies with local dampers (such as discrete or viscous-ring) of appropriate parameters, and so on. We do not assume that the kinetic energy of a system of quasirigid bodies decreases because we allow the presence of energy sources. Thus, our model of a dual spinner (axisymmetric) consists of a rigid rotor and a rigid platform (see Fig. 1), each of which contains energy sink devices (not shown in the figure), hereafter referred to simply as dampers. It also contains a motor (a possible energy source, also not shown in the figure) whose parts belong to the rotor or the platform.

The assumptions of quasirigidity and zero net external torque on the spacecraft allow us to use an expression for the angular momentum that is based on a rigid-body system,[16]

$$H^2 = I_t^2\omega_t^2 + (I_s\omega_s + J\Omega)^2 \tag{4}$$

as an integral of motion. Likewise, the kinetic energy of the dual spinner can be approximated by[16]

$$2E = I_t\omega_t^2 + I_s\omega_s^2 + J\Omega^2 + 2J\Omega\omega_s \tag{5}$$

The approximate nature of these equations suggests that their time derivatives represent an average rate of change of their true values.

It is important to recognize that the definition of quasirigidity does not allow us to use Euler's rigid-body equations[1] (for example, $T^*_{P/M} \neq I_{s,P}\dot{\omega}_s$). Therefore, we introduce a separation axiom to derive an expression for the motor torque on a quasirigid body.

Separation Axiom

Consider an interim model of a dual spinner consisting of a rigid, axisymmetric rotor and a quasirigid platform (i.e., $\dot{E}_R = 0$). Thus, although $T^*_{P/M} \neq I_{s,P}\dot{\omega}_s$, we can write

$$T_R = T_{R/M} = J(\dot{\Omega} + \dot{\omega}_s) \tag{6}$$

and from Newton's third law of motion

$$T_{R/M} + T_{P/M} = 0 \tag{7}$$

Eq. (6) may be rewritten as

$$T_{P/M} = -J(\dot{\Omega} + \dot{\omega}_s) \tag{8}$$

This innocuous expression becomes more powerful when the right-hand side of the equation is expressed solely in terms of the platform variables. This is done in two steps. First, from (see Fig. 1)

$$\cos\eta = \frac{H_s}{H} = \frac{J\Omega + I_s\omega_s}{H} \tag{9}$$

we get

$$\dot{\eta}H\sin\eta = -(J\dot{\Omega} + I_s\dot{\omega}_s) \tag{10}$$

Combining Eqs. (8) and (10), we arrive at

$$T_{P/M} = I_{s,P}\dot{\omega}_s + \dot{\eta}H\sin\eta \tag{11}$$

where we have made use of $I_s = I_{s,P} + J$. The second step is to relate the last term of the right-hand side of this equation to the energy dissipation in the platform, $\dot{E}_P$. To do this, we separate the (possible) energy source from the energy sink by writing

$$\dot{E}_P = \dot{E} - \dot{W} \tag{12}$$

where $\dot{E}$ is determined by differentiating Eq. (5),

$$\dot{E} = I_t\omega_t\dot{\omega}_t + I_s\omega_s\dot{\omega}_s + J\dot{\Omega}\omega_s + J(\dot{\Omega} + \dot{\omega}_s)\Omega \tag{13}$$

whereas $\dot{W}$ is obtained directly from its definition,

$$\dot{W} = T_{R/M}\Omega = J(\dot{\Omega} + \dot{\omega}_s)\Omega \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (12) yields

$$\dot{E}_P = I_t\omega_t\dot{\omega}_t + (I_s\dot{\omega}_s + J\dot{\Omega})\omega_s \tag{15}$$

Differentiating Eq. (4) we have

$$I_t^2\omega_t\dot{\omega}_t + (I_s\omega_s + J\Omega)(I_s\dot{\omega}_s + J\dot{\Omega}) = 0 \tag{16}$$

The rate of change of the transverse angular velocity component may be eliminated from Eqs. (15) and (16) and simplified to

$$\dot{E}_P = -(\dot{\omega}_s I_s + J\dot{\Omega})\lambda_P \tag{17}$$

where we have substituted for the platform nutation frequency defined by

$$\lambda_P = \frac{\omega_s(I_s - I_t) + J\Omega}{I_t} \tag{18}$$

Comparing Eqs. (17) and (10), it is clear that

$$\dot{\eta}H\sin\eta = \dot{E}_P/\lambda_P \tag{19}$$

Thus, Eq. (11) reduces to

$$T_{P/M} = I_{s,P}\dot{\omega}_s + (\dot{E}_P/\lambda_P) \tag{20}$$

We now have the sought expression for the motor torque on the quasirigid platform solely in terms of the platform variables. It may be interpreted as an application of Euler's equations to the rigid part of the quasirigid platform with $-\dot{E}_P/\lambda_P$ as a measure of the damper's reaction torque. It is noteworthy that this is not how it was obtained.

Similarly, for a quasirigid rotor and a rigid platform, we can write

$$T_{R/M} = J(\dot{\Omega} + \dot{\omega}_s) + (\dot{E}_R/\lambda_R) \tag{21}$$

where the rotor nutation frequency is given by $\lambda_R = \lambda_P - \Omega$.

Now suppose that $T^*_{P/M}$ and $T^*_{R/M}$ are the motor torques on the platform and the rotor, respectively, when both bodies are quasirigid. We propose a separation axiom in which we insist that

$$T^*_{P/M} = (\dot{E}_P/\lambda_P) + I_{s,P}\dot{\omega}_s \tag{22a}$$

$$T^*_{R/M} = (\dot{E}_R/\lambda_R) + J(\dot{\Omega} + \dot{\omega}_s) \tag{22b}$$

Note that we have not introduced an additional axiom to the classical energy-sink analysis; rather, this is really a refined version of a separation axiom used by other investigators,[11–15] who show that

$$\dot{E} = -I_{s,P}\dot{\omega}_s\lambda_P - J(\dot{\omega}_s + \dot{\Omega})\lambda_R \tag{23}$$

and then by comparing it to Eq. (2) separate it into two equations,

$$\dot{E}_P = -I_{s,P}\dot{\omega}_s\lambda_P \tag{24a}$$

$$\dot{E}_R = -J(\dot{\omega}_s + \dot{\Omega})\lambda_R \tag{24b}$$

As shown here by Eqs. (22), this is true only if the motor is absent, or effectively absent by providing just enough torque to compensate for the bearing friction. This hitherto restricted the application of the Landon–Iorillo stability criterion since, in many practical situations, the relative rotor speed Ω is held constant by a simple feedback control system, thus altering (and possibly increasing) the kinetic energy of the spacecraft.

Analysis

The Landon–Iorillo stability criterion follows very simply from Newton's third law of motion

$$T^*_{R/M} + T^*_{P/M} = 0 \tag{25}$$

and Eqs. (22) to yield

$$(\dot{E}_P/\lambda_P) + (\dot{E}_R/\lambda_R) = -(I_s\dot{\omega}_s + J\dot{\Omega}) = \dot{\eta}H\sin\eta \tag{26}$$

where the second equality follows from Eq. (10). For stability, $\dot{\eta} \le 0 \Rightarrow$ Eq. (1), a classical result shown to be true despite the presence of a motor torque and possible energy additions. A consequence of this analysis is that it allows us to determine the sign definiteness of energy. Eliminating the transverse component of the angular velocity from Eqs. (4) and (5), the resulting equation may be solved for ω_s and substituted in Eq. (9). This yields (for $0 \le \eta \le \pi/2$) (Ref. 5)

$$\cos\eta = \sqrt{Q} \tag{27a}$$

where

$$Q = \frac{(2E - J\Omega^2)I_s + (J\Omega)^2}{H^2(1 - I_s/I_t)} - \frac{I_s}{I_t - I_s} \tag{27b}$$

Differentiating this equation and simplifying,

$$\left(\frac{H^2\sqrt{Q}\sin\eta}{I_{s,P}}\right)\dot{\eta} = \frac{-\dot{E}(I_s/I_{s,P}) + \dot{\Omega}J\Omega}{(1 - I_s/I_t)} \tag{28}$$

shows that the sign of $\dot{E}$ depends on a number of factors and, therefore, cannot be assumed a priori. As was explained earlier, much of the controversy arose because of the application of the energy-sink theory to the special case, Ω = const, which is how many practical dual spinners operate.[6–9] For this case, it is clear from Eq. (28) that energy is maximal ($\dot{E} \ge 0$) for a stable ($\dot{\eta} \le 0$), prolate ($I_s/I_t < 1$) dual-spin spacecraft. Thus, an assumption of $\dot{E} \le 0$ (the old energy-sink axiom) for the condition that Ω be a constant is self-contradictory.

An alternative set of stability criteria may be obtained for the special case, Ω = constant, by way of Hubert's core energy. Following Hubert,[6,7] we define the platform core energy of a dual spinner as

$$2E_{c,P} = I_t\omega_t^2 + I_s\omega_s^2 \tag{29}$$

Physically, it represents the kinetic energy of a hypothetical rigid body possessing the inertia properties of the dual spinner but spinning like the platform. In like manner, we can define the rotor core energy as

$$2E_{c,R} = I_t\omega_t^2 + I_s(\omega_s + \Omega)^2 \tag{30}$$

With these definitions, one can write Eq. (5) as either

$$2E = 2E_{c,P} + J\Omega^2 + 2J\Omega\omega_s \tag{31}$$

or

$$2E = 2E_{c,R} - I_{s,P}\Omega^2 - 2I_{s,P}\Omega\omega_s \tag{32}$$

Separating the possible energy source from the energy sinks, we get

$$\dot{E}_D = \dot{E} - \dot{W} \tag{33}$$

where, by definition,

$$\dot{E}_D = \dot{E}_P + \dot{E}_R \tag{34}$$

$$\dot{W} = T^*_{R/M}\Omega \tag{35}$$

From Eqs. (22), (31), (33), and (34), it is straightforward to show that

$$(\dot{E}_P/\lambda_P) + (\dot{E}_R/\lambda_R) = (\dot{E}_{c,P}/\lambda_P) + (J\dot{\Omega}\omega_s/\lambda_P) \tag{36}$$

where we have used the relation $\lambda_P = \lambda_R + \Omega$. Thus, when $\dot{\Omega} = 0$, we have

$$(\dot{E}_P/\lambda_P) + (\dot{E}_R/\lambda_R) = \dot{E}_{c,P}/\lambda_P \tag{37}$$

From symmetry, it follows that

$$(\dot{E}_P/\lambda_P) + (\dot{E}_R/\lambda_R) = \dot{E}_{c,R}/\lambda_R \tag{38}$$

which can also be proved independently from Eqs. (22) and (32–34). Thus, we arrive at a symmetric stability condition,

$$\dot{E}_{c,P}/\lambda_P = \dot{E}_{c,R}/\lambda_R \le 0 \tag{39}$$

from which it is clear that, for stability, the core energies (platform or rotor) are either minimal or maximal, depending on the sign of the (platform or rotor) nutation frequencies. Numerical simulations corroborate this result.[17]

Finally, from Eq. (36), we note that

$$\dot{E}_{c,P} = \dot{E}_P + \dot{E}_R(\lambda_P/\lambda_R) - J\dot{\Omega}\omega_s \tag{40}$$

Hence, for the platform core energy to be minimal, it is necessary and sufficient that

$$J\dot{\Omega}\omega_s > \dot{E}_P + \dot{E}_R(\lambda_P/\lambda_R) \tag{41}$$

which generalizes previous results.[6–8,10]

Conclusions

The energy-sink theory may be viewed as a tool to solve certain stability problems for systems that contain unmodeled or generic energy-dissipating devices. In this paper, the theory was strengthened by deriving the Landon–Iorillo stability criterion without the old energy-sink axiom, viz., the assumption of total energy dissipation ($\dot{E} \le 0$). In fact, contradictions arise when the sign of $\dot{E}$ is presumed to be negative or positive for a system containing a driven rotor. A natural consequence of applying Landon's old idea using the relatively new concept of Hubert's core energy results in a symmetric stability criterion for the special case of a constant relative spin speed of the rotor. This result clarifies Hubert's ideas on minimizing the core energy and generalizes it to systems that contain quasirigid rotors as well. The new stability criterion may be used to postulate the behavior of the core energy and, thus, quantitatively predict the nutation angle by use of the core-energy integral.

Acknowledgments

The author gratefully acknowledges funding provided for this research by the U.S. Naval Postgraduate School. The author would also like to thank Carl Hubert, whose comments immensely helped in revising the manuscript.

References

[1] Landon, V. D., and Stewart, B., "Nutational Stability of an Axisymmetric Body Containing a Rotor," *Journal of Spacecraft and Rockets*, Vol. 1, No. 6, 1964, pp. 682–684.

[2]Likins, P. W., "Attitude Stability Criteria for Dual Spin Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 4, No. 12, 1967, pp. 1638–1643.

[3]Cherchas, D. B., and Hughes, P. C., "Attitude Stability of a Dual-Spin Satellite with a Large Flexible Solar Array," *Journal of Spacecraft and Rockets*, Vol. 10, No. 2, 1973, pp. 126–132.

[4]Spencer, T. M., "Energy-Sink Analysis for Asymmetric Dual-Spin Spacecraft," *Journal of Spacecraft and Rockets*, Vol. 11, No. 7, 1974, pp. 463–468.

[5]Kane, T. R., and Levinson, D. A., "Energy-Sink Analysis of Systems Containing Driven Rotors," *Journal of Guidance and Control*, Vol. 3, No. 3, 1980, pp. 234–238.

[6]Hubert, C., "The Use of Energy Methods in the Study of Dual-Spin Spacecraft," *Proceedings of the AIAA Guidance and Control Conference* (Danvers, MA), AIAA, New York, 1980, pp. 372–375 (AIAA Paper 80-1781).

[7]Hubert, C., "Spacecraft Attitude Acquisition from an Arbitrary Spinning or Tumbling State," *Journal of Guidance and Control*, Vol. 4, No. 2, 1981, pp. 164–170.

[8]Cochran, J. E., and Shu, P. H., "Effects of Energy Addition and Dissipation on Dual-Spin Spacecraft Attitude Motion," *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 5, 1983, pp. 368–373.

[9]Chung, T., and Kane, T. R., "Energy Sink Analysis of Dual Spin Spacecraft," INTEL Rept. No. 465, INTELSAT, Washington, DC, Oct. 1985.

[10]Ross, I. M., "Nutational Stability and Core Energy of a Quasirigid Gyrostat," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 4, 1993, pp. 641–647.

[11]Likins, P. W., "Spacecraft Attitude Dynamics and Control—A Personal Perspective on Early Developments," *Journal of Guidance, Control, and Dynamics*, Vol. 9, No. 2, 1986, pp. 129–134.

[12]Agrawal, B. N., *Design of Geosynchronous Spacecraft*, Prentice–Hall, Englewood Cliffs, NJ, 1986, Chap. 3.

[13]Chobotov, V. A., *Spacecraft Attitude Dynamics and Control*, Krieger, Malabar, FL, 1991, Chap. 3.

[14]Hughes, P. C., *Spacecraft Attitude Dynamics*, Wiley, New York, 1986, Chaps. 5 and 7.

[15]Kaplan, M. H., *Modern Spacecraft Dynamics and Control*, Wiley, New York, 1976, Chap. 5.

[16]Kane, T. R., Likins, P. W., and Levinson, D. A., *Spacecraft Dynamics*, McGraw–Hill, New York, 1983, Chap. 3.

[17]Ortiz, V. M., "Evaluation of Energy-Sink Stability Criteria for Dual-Spin Spacecraft," M.S. Thesis, Dept. of Physics, U.S. Naval Postgraduate School, Monterey, CA, June 1994.